Problem session (Geometric and analytic approaches to representations of a group and representation spaces)

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数理解析研究所講究録
第 1777 巻 2012 年 110-112

Problem session

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I. Finite representations of knot groups.

The method of mapping knot groups onto finite groups is a very effective method for distinguishing the groups (see [10, 11, 3]). So, it is natural to ask if this method is always successful at distinguishing the groups (see [11, Page 30]).

Problem 1 (1) Can we distinguish knot groups by counting the numbers of transitive representations of the knot groups to the symmetric group S_n of degree n? To be precise, for a knot group G and a positive integer n, let $R(G;n)$ be the set of transitive representations of G to S_n modulo post composition of inner automorphisms of S_n. Then its cardinality $|R(G;n)|$ is of course an invariant of the knot group. Is the family of invariants, $\{|R(G;n)|\}_n$, a complete invariant of the knot group? Namely, for two non-isomorphic knot groups G_1 and G_2, can we always find a positive integer n such that $|R(G_1;n)|\neq|R(G_2;n)|$?

(2) When a meridian, μ, of G is specified, we can refine $R(G;n)$ as follows. Let $(n_1,n_2,\cdots,n_k)$ be a sequence of positive integers such that $n_1+n_2+\cdots+n_k=n$ and $n_1\leq n_2\leq\cdots\leq n_k$. Let $R(G,\mu;n_1,n_2,\cdots,n_k)$ be the subset of $R(G;n)$ consisting of those representations which map μ to a product of mutually disjoint cyclic permutations of length $n_1,n_2,\ldots,n_k$. Then is the family of the invarinats, $\{|R(G,\mu;n_1,n_2,\cdots,n_k)|\}$, a complete invariant of (G,μ)?

(3) We can also consider the homology of branched/unbranched coverings associated with transitive representations of G to finite symmetric groups. Is the combination of the invariants $\{|R(G;n)|\}_n$ (resp. $\{|R(G,\mu;n_1,n_2,\cdots,n_k)|\}$) and the homology of associate finite branched/unbranched coverings a complete invariant of G (resp. (G,μ))?

Remark 2 In [3], we had to distinguish various pairs of mutants, and this was carried out by using the above methods with the help of Kodama's software [2].

Problem 1 motivates the following problem.

Problem 3 Is it true that two non-isomorphic knot groups have non-isomorphic profinite completions?

II. Simple loops on bridge spheres.

We present variations of the problems on Heegaard splittings of 3-manifolds raised by Y. Minsky [1, Question 5.4]. For a knot K in the 3-sphere S^3, let $(S^3,K)=(B_1^3,t_1)\cup(B_2^3,t_2)$ be an n-bridge decomposition of K and set $S:=\partial B_1^3\setminus t_1(=\partial B_2^3\setminus t_2)$.

Problem 4 (1) Which essential simple loop in S is null-homotopic in $S^3\setminus K$?

(2) Which essential simple loops in S are mutually homotopic in $S^3\setminus K$?

Let $\mathcal{M}(S)$ and $\mathcal{M}(B_i^3,t_i)$ $(i=1,2)$, respectively, be the mapping class groups $\pi_0\mathrm{Diff}(S)$ and $\pi_0\mathrm{Diff}(B_i^3,t_i)$. For each $i=1,2$, let $\mathcal{M}_0(B_i^3,t_i)$ be the subgroup of

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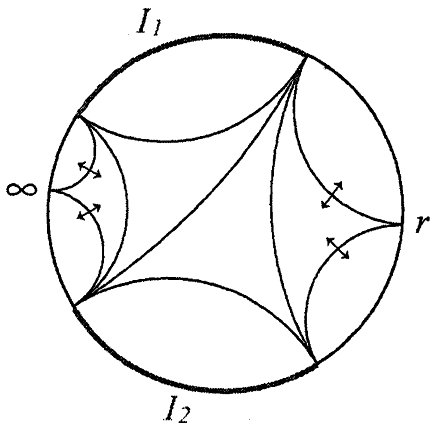

Figure 1:

$\mathcal{M}(B_i^3, t_i)$ which consists of elements which induce the identity element in the outer-automorphism group $\mathrm{Out}(\pi_1(B_i^3 \setminus t_i))$. Let Γ be the subgroup of $\mathcal{M}(S)$ generated by $\mathcal{M}_0(B_1^3, t_1) \cup \mathcal{M}_0(B_2^3, t_2)$. Let Δ_i $(i = 1, 2)$ be the set of essential simple loops in S which bounds a disk in $B_i^3 \setminus t_i$, and let Δ be the union of Δ_1 and Δ_2. Note that Δ is a subcomplex of the curve complex $\mathcal{C}^{(0)}(S)$ of S.

Observation 5 *Any simple loop in $\Gamma\Delta$ is null-homotopic.*

Problem 6 Is the converse true if the bridge decomposition is "complicated enough"?

Let $\mathcal{PML}(S)$ be the projective measured lamination space of S. Though the action of $\mathcal{M}(S)$ on $\mathcal{PML}(S)$ is ergodic, the action of $\mathcal{M}_0(B_i^3, t_i)$ on $\mathcal{PML}(S)$ would have a non-empty domain of discontinuity for each $i = 1, 2$ (see Masur [9]).

Problem 7 If the bridge decomposition of K is "complicated enough", then does the action of $\Gamma(\subset \mathcal{M}(S))$ on $\mathcal{PML}(S)$ have a nonempty domain of discontinuity?

Problem 8 Is Γ isomorphic to the free product of $\mathcal{M}_0(B_1^3, t_1)$ and $\mathcal{M}_0(B_2^3, t_2)$?

Problem 9 Let $\Omega(\Gamma)$ be the domain of discontinuity of the action of Γ on $\mathcal{PML}(S)$. If a loop c on S belongs to the intersection of $\Omega(\Gamma)$ and $\mathcal{C}^{(0)}(S)$, then is c not null-homotopic in $S^3 \setminus K$?

Problem 10 Can we find an open set U in $\mathcal{PML}(S)$ such that any loop which belongs to the intersection of U and $\mathcal{C}^{(0)}(S)$ is not null-homotopic in $S^3 \setminus K$?

Let Δ^* be the closure in $\mathcal{PML}(S)$ of the set of loops in $\mathcal{C}^{(0)}(S)$ which is null-homotopic in $S^3 \setminus K$.

Problem 11 Does Δ^* have measure 0?

Remark 12 For 2-bridge spheres of 2-bridge links, Problems 4 - 11 are solved affirmatively (see [4, 5, 6, 7, 8]). In particular, for a 2-bridge link $K(r)$, the action of Γ on $\mathcal{PML}(S)$ has the domain of discontinuity, and the union of two intervals $I_1 \cup I_2$ in Figure 1 forms a fundamental domain.

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